ESD-TR-65-112

SR-57

DYNAMIC STRESSES IN AN ELASTIC CYLINDRICAL LINING OF ARBITRARY THICKNESS IN AN ELASTIC MEDIUM

TECHNICAL DOCUMENTARY REPORT NO. ESD-TR-65-112

APRIL 1965

C. C. Mow

W. L. McCabe

Prepared for

DIRECTOR OF ANALYSIS
DEPUTY FOR ADVANCED PLANNING
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 607
Prepared by

THE MITRE CORPORATION
Bedford, Massachusetts

Contract AF33(600)39852

ESD-TR-65-112 SR-57

DYNAMIC STRESSES IN AN ELASTIC CYLINDRICAL LINING OF ARBITRARY THICKNESS IN AN ELASTIC MEDIUM

TECHNICAL DOCUMENTARY REPORT NO. ESD-TR-65-112

APRIL 1965

C. C. Mow

W. L. McCabe

Prepared for

DIRECTOR OF ANALYSIS
DEPUTY FOR ADVANCED PLANNING
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 607
Prepared by

THE MITRE CORPORATION
Bedford, Massachusetts

Contract AF33(600)39852

FOREWORD

The authors wish to thank Charles McCarthy of The MITRE Corporation for obtaining the numerical results, and Prof. Y.H. Pao of Cornell University for his advice during the course of the study.

DYNAMIC STRESSES IN AN ELASTIC CYLINDRICAL LINING OF ARBITRARY THICKNESS IN AN ELASTIC MEDIUM

ABSTRACT

Dynamic stresses in a thick-wall elastic cylinder in an infinite elastic medium during passage of plane, compressional waves are investigated. Dynamic stresses around the cylinder in the elastic medium are also determined. Numerical results for two different cylinders with ratios of outer radius to inner radius ranging from 1.05 to 1.20 are presented in a dimensionless form. It is shown that increasing thickness does not, in general, reduce stresses in the cylinder; in addition, dynamic stresses at certain wave numbers are higher than the corresponding static value.

REVIEW AND APPROVAL

This technical documentary report has been reviewed and is approved.

Francis J. Dillon Jr.

FRANCIS J. DILLON, JR.
Col., USAF
Director of Analysis
Deputy for Advanced Planning

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Continued)

SECTION I

INTRODUCTION

The problem of dynamic stresses in an infinite medium containing cavities, rigid inclusions, and elastic inclusions has been studied extensively[1, 2, 3, 4] the problem of scattering or diffraction of sound or stress waves by thin elastic shells in fluid and elastic media has also been the subject of many investigations.[5, 6, 7]

The problem presented herein is the response of a thick-wall cylinder in an infinite elastic medium subjected to a progressing compressional wave. It is assumed that the thick-wall cylinder is of infinite extent and is embedded in and bound to an infinite elastic medium. A plane compressional wave of harmonic time variation propagates in the positive X-direction (Fig. 1) and impinges on the embedded cylinder.

It is known that if the material constants (λ, μ, and ρ) of the cylinder and the elastic medium are different, scattering will occur at the interface of the cylinder and the surrounding medium. If, however, the material constants are the same, the problem should reduce to the simple cavity case presented in reference 1.

In general, at the boundary of the cylinder and the surrounding medium, the incident compressional waves are reflected and refracted as compressional and shear waves. Hence, there will be two refracted waves propagating into the cylinder. These waves will be reflected at the traction-free boundary of the inner surface of the cylinder. Therefore, seven waves exist; one incident wave, two reflected waves in the elastic medium, and four in the cylinder due to refraction and reflection.

The solution of this problem involves finding the coefficients associated with the six unknown waves. This is accomplished by four equations of continuity at the interface and two boundary conditions at the traction-free inner surface of the cylinder.

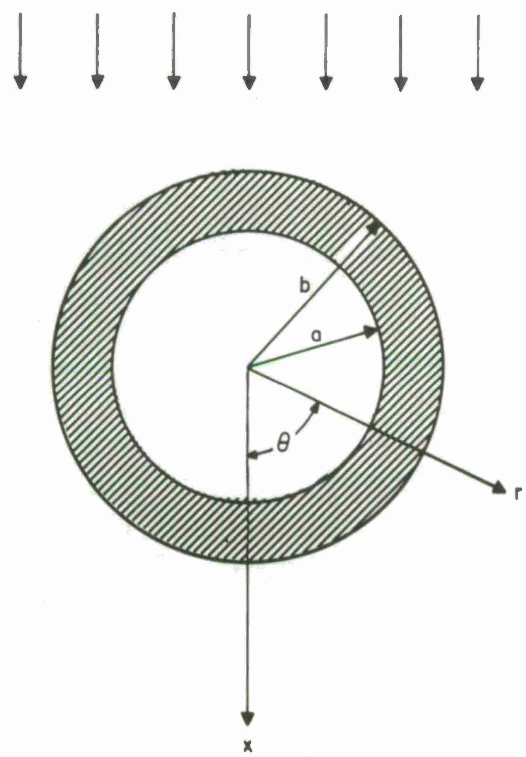

Fig. 1 Cylindrical Lining

SECTION II

GENERAL THEORY

If it is assumed that the cylinder is infinite in extent, the problem becomes one of generalized plane strain. The displacement equation of motion is

$$(\lambda + \mu)\nabla\nabla \cdot \underset{\sim}{u} + \mu \nabla^2 \underset{\sim}{u} = \rho \ddot{\underset{\sim}{u}} \tag{1}$$

where

- $\underset{\sim}{u}$ is the displacement vector
- ∇ is the gradient operator
- λ and μ are the Lamé Constants
- ρ is the density.

The displacement vector $\underset{\sim}{u}$ can be represented by a scalar potential and a vector potential; in the case of plane strain this is

$$\underset{\sim}{u} = \nabla \phi + \nabla \mathrm{x} (\underset{\sim}{e}_z \psi) \tag{2}$$

Each potential function then satisfies a scalar wave equation

$$c_\alpha^2 \nabla^2 \phi = \ddot{\phi} \tag{3}$$

$$c_\beta^2 \nabla^2 \psi = \ddot{\psi} \tag{4}$$

In Equations (2), (3), and (4), $\underset{\sim}{e}_z$ is a unit vector along the axis of the cylinder and

$$c_\alpha^2 = \frac{\lambda + 2\mu}{\rho} \qquad c_\beta^2 = \frac{\mu}{\rho}$$

As shown in Fig. 1, in plane polar coordinates (r, θ) the scalar form of Equation (2) is

$$u_r = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} - \frac{\partial \psi}{\partial r} \tag{5}$$

and the stresses are related to the potentials by

$$\tau_{rr} = \lambda \nabla^2 \phi + 2\mu\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \psi}{\partial \theta}\right)$$

$$\tau_{\theta\theta} = \lambda \nabla^2 \phi + 2\mu\left(\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \psi}{\partial \theta}\right)$$

$$\tau_{r\theta} = 2\mu\left(\frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} - \frac{1}{2}\nabla^2 \psi\right) \tag{6}$$

SECTION III

INCIDENT REFLECTED AND REFRACTED WAVES

For convenience, the surrounding elastic medium and the thick-wall cylinder will be denoted as regions Nos. 1 and 2, respectively.

The incident wave, propagating in the positive X-direction is represented by

$$\phi^{(i)}_{(1)} = \phi_0 \, e^{i(\alpha_1 x - \omega t)}$$

$$\psi^{(i)}_{(1)} = 0 \tag{7}$$

where

ϕ_0 is a measure of the amplitude

ω the circular frequency

$\alpha_1 = \omega / c_{\alpha_1}$ is the wave number of the compressional wave.

In polar coordinates

$$\phi^{(i)}_{(1)} = \phi_0 \sum_{n=0}^{\infty} \epsilon_n \, i^n \, J_n(\alpha_1 r) \cos n\theta \, e^{-i\omega t} \tag{8}$$

where

J_n denotes the Bessel function of the first kind of order n

ϵ_n is a constant so that

$$\epsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n \geq 1 \end{cases}$$

Let the origin of the polar coordinates coincide with the central axis of the cylinder. The waves in regions Nos. 1 and 2 can then be expressed as follows.

REGION NO. 1

$$\phi^{(R)}_{(1)} = \sum_{n=0}^{\infty} A_n H_n^{(1)} (\alpha_1 r) \cos n\theta \, e^{-i\omega t} \tag{9}$$

$$\psi^{(R)}_{(1)} = \sum_{n=0}^{\infty} B_n H_n^{(1)} (\beta_1 r) \sin n\theta \, e^{-i\omega t} \tag{10}$$

REGION NO. 2

$$\phi^{(i)}_{(2)} = \sum_{n=0}^{\infty} M_n H_n^{(2)} (\alpha_2 r) \cos n\theta \, e^{-i\omega t} \tag{11}$$

$$\psi^{(i)}_{(2)} = \sum_{n=0}^{\infty} N_n H_n^{(2)} (\beta_2 r) \sin n\theta \, e^{-i\omega t} \tag{12}$$

$$\phi^{(R)}_{(2)} = \sum_{n=0}^{\infty} R_n H_n^{(1)} (\alpha_2 r) \cos n\theta \, e^{-i\omega t} \tag{13}$$

$$\psi^{(R)}_{(2)} = \sum_{n=0}^{\infty} S_n H_n^{(1)} (\beta_2 r) \sin n\theta \, e^{-i\omega t} \tag{14}$$

In Equations (9) through (14)

$\phi^{(R)}_{(1)}$ $\psi^{(R)}_{(1)}$ represents the compressional and shear waves in region No. 1

$\phi^{(i)}_{(2)}$ $\psi^{(i)}_{(2)}$ represents the inward propagating compressional and shear waves in region No. 2

$\phi^{(R)}_{(2)}$ $\psi^{(R)}_{(2)}$ are the outgoing waves in region No. 2

A_n, B_n, M_n, N_n, R_n, and S_n are expansion coefficients to be determined

α_1 and α_2 are the compressional wave numbers in regions Nos. 1 and 2

β_1 and β_2 are the shear wave numbers in regions Nos. 1 and 2 ($\beta_1 = \frac{\omega}{c_{\beta_1}}$ etc.)

$H_n^{(1)}$ and $H_n^{(2)}$ denote the Hankel functions of the first and second kind of order n. The Hankel function of the first kind is used for diverging waves; e.g., in Equations (9), (10), (13), and (14). The Hankel function of the second kind represents converging waves; e.g., in Equations (11) and (12).

In Equations (9) through (14) there are six undetermined coefficients. The boundary conditions which allow these coefficients to be determined are listed below.

At r = b the condition of continuity requires that

$$
\begin{aligned}
\tau_{rr_{(2)}} &= \tau_{rr_{(1)}} \\
\tau_{r\theta_{(2)}} &= \tau_{r\theta_{(1)}} \\
u_{r_{(2)}} &= u_{r_{(1)}} \\
u_{\theta_{(2)}} &= u_{\theta_{(1)}}
\end{aligned}
\tag{15}
$$

At r = a the traction-free boundary implies

$$
\begin{aligned}
\tau_{rr_{(2)}} &= 0 \\
\tau_{r\theta_{(2)}} &= 0
\end{aligned}
\tag{16}
$$

SECTION IV

SOLUTIONS

Where $\tau_{rr_{(1),(2)}}$ $\tau_{r\theta_{(1),(2)}}$ $u_{r_{(1),(2)}}$ $u_{\theta_{(1),(2)}}$ are the stresses and displacements due to the total displacement potential in regions Nos. 1 and 2; e.g.,

$$\tau_{rr_{(1)}} = \tau_{rr_{(1)}}\left(\phi_{(1)}\ \psi_{(1)}\right)$$

$$\tau_{rr_{(2)}} = \tau_{rr_{(2)}}\left(\phi_{(2)}\ \psi_{(2)}\right) \qquad (17)$$

where

$$\phi_{(1)} = \phi^{(i)}_{(1)} + \phi^{(R)}_{(1)}$$

$$\psi_{(1)} = \psi^{(R)}_{(1)}$$

$$\phi_{(2)} = \phi^{(i)}_{(2)} + \phi^{(R)}_{(2)}$$

$$\psi_{(2)} = \psi^{(i)}_{(2)} + \psi^{(R)}_{(2)} \qquad (18)$$

Substituting Equations (9) through (14) and (18) into Equations (5) and (6) yields the corresponding displacement and stress components in regions Nos. 1 and 2. With the time factor $e^{-i\omega t}$ omitted, the expressions for the displacement are

$$u_{r_{(1)}} = r^{-1} \sum_{n=0}^{\infty} \left[\phi_0 \epsilon_n i^n \alpha_1 r J_n'(\alpha_1 r) + A_n \alpha_1 r H_n^{(1)'}(\alpha_1 r) + B_n n H_n^{(1)}(\beta_1 r) \right] \cos n\theta \tag{19}$$

$$u_{\theta_{(1)}} = -r^{-1} \sum_{n=0}^{\infty} \left[\phi_0 \epsilon_n i^n n J_n(\alpha_1 r) + A_n n H_n^{(1)}(\alpha_1 r) + B_n \beta_1 r H_n^{(1)'}(\beta_1 r) \right] \sin n\theta \tag{20}$$

$$u_{r_{(2)}} = r^{-1} \sum_{n=0}^{\infty} \Big[M_n \alpha_2 r H_n^{(2)'}(\alpha_2 r) + N_n n H_n^{(2)}(\beta_2 r) + R_n \alpha_2 r H_n^{(1)'}(\alpha_2 r) + S_n n H_n^{(1)}(\beta_2 r) \Big] \cos n\theta \tag{21}$$

$$u_{\theta_{(2)}} = -r^{-1} \sum_{n=0}^{\infty} \Big[M_n n H_n^{(2)}(\alpha_2 r) + N_n \beta_2 r H_n^{(2)'}(\beta_2 r) + R_n n H_n^{(1)}(\alpha_2 r) + S_n \beta_2 r H_n^{(1)'}(\beta_2 r) \Big] \sin n\theta \tag{22}$$

The expressions for the stress are

$$\tau_{rr_{(1)}} = 2\mu_1 r^{-2} \sum_{n=0}^{\infty} \left(\phi_0 \epsilon_n i^n {}_1D_{nr}^{(i)} + A_n {}_1D_{nr}^{(R)} + B_n {}_1\mathfrak{D}_{nr}^{(R)} \right) \cos n\theta \tag{23}$$

$$\tau_{\theta\theta_{(1)}} = 2\mu_1 r^{-2} \sum_{n=0}^{\infty} \left(\phi_0 \epsilon_n i^n {}_1F_{nr}^{(i)} + A_n {}_1F_{nr}^{(R)} - B_n {}_1\mathfrak{D}_{nr}^{(R)} \right) \cos n\theta \tag{24}$$

$$\tau_{r\theta_{(1)}} = 2\mu_1 r^{-2} \sum_{n=0}^{\infty} (\phi_0 \epsilon_n i^n \, {}_1E_{nr}^{(i)} + A_{n1} E_{nr}^{(R)} + B_{n1} \mathcal{E}_{nr}^{(R)}) \sin n\theta \tag{25}$$

$$\tau_{rr_{(2)}} = 2\mu_2 r^{-2} \sum_{n=0}^{\infty} (M_{n2} D_{nr}^{(i)} + N_{n2} \mathfrak{D}_{nr}^{(i)} + R_{n2} D_{nr}^{(R)} + S_{n2} \mathfrak{D}_{nr}^{(R)}) \cos n\theta \tag{26}$$

$$\tau_{\theta\theta_{(2)}} = 2\mu_2 r^{-2} \sum_{n=0}^{\infty} (M_{n2} F_{nr}^{(i)} - N_{n2} \mathfrak{D}_{nr}^{(i)} + R_{n2} F_{nr}^{(R)} - S_{n2} \mathfrak{D}_{nr}^{(R)}) \cos n\theta \tag{27}$$

$$\tau_{r\theta_{(2)}} = 2\mu_2 r^{-2} \sum_{n=0}^{\infty} (M_{n2} E_{nr}^{(i)} + N_{n2} \mathcal{E}_{nr}^{(i)} + R_{n2} E_{nr}^{(R)} + S_{n2} \mathcal{E}_{nr}^{(R)}) \sin n\theta \tag{28}$$

with

$${}_1D_{nr}^{(i)} = (n^2 + n - \frac{1}{2} \beta_1^2 r^2) J_n(\alpha_1 r) - \alpha_1 r J_{n-1}(\alpha_1 r)$$

$${}_1E_{nr}^{(i)} = (n^2 + n) J_n(\alpha_1 r) - n \alpha_1 r J_{n-1}(\alpha_1 r)$$

$${}_1F_{nr}^{(i)} = -(n^2 + n - \alpha_1^2 r^2 + \frac{1}{2} \beta_1^2 r^2) J_n(\alpha_1 r) + \alpha_1 r J_{n-1}(\alpha_1 r) \tag{29}$$

$$ {}_j D^{(R)}_{nr} = (n^2 + n - \frac{1}{2} \beta_j^2 r^2) H_n^{(1)}(\alpha_j r) - \alpha_j r H_{n-1}^{(1)}(\alpha_j r) $$

$$ {}_j E^{(R)}_{nr} = (n^2 + n) H_n^{(1)}(\alpha_j r) - n\alpha_j r H_{n-1}^{(1)}(\alpha_j r) $$

$$ {}_j F^{(R)}_{nr} = -(n^2 + n - \alpha_j^2 r^2 + \frac{1}{2} \beta_j^2 r^2) H_n^{(1)}(\alpha_j r) + \alpha_j r H_{n-1}^{(1)}(\alpha_j r) \quad (30) $$

$$ j = 1, 2 $$

$$ {}_2 D^{(i)}_{nr} = (n^2 + n - \frac{1}{2} \beta_2^2 r^2) H_n^{(2)}(\alpha_2 r) - \alpha_2 r H_{n-1}^{(2)}(\alpha_2 r) $$

$$ {}_2 E^{(i)}_{nr} = (n^2 + n) H_n^{(2)}(\alpha_2 r) - n\alpha_2 r H_{n-1}^{(2)}(\alpha_2 r) $$

$$ {}_2 F^{(i)}_{nr} = -(n^2 + n - \alpha_2^2 r^2 + \frac{1}{2} \beta_2^2 r^2) H_n^{(2)}(\alpha_2 r) + \alpha_2 r H_{n-1}^{(2)}(\alpha_2 r) \quad (31) $$

$$ {}_j \mathfrak{D}^{(R)}_{nr} = -(n^2 + n) H_n^{(1)}(\beta_j r) + n\beta_j r H_{n-1}^{(1)}(\beta_j r) $$

$$ {}_j \mathcal{E}^{(R)}_{nr} = -(n^2 + n - \frac{1}{2} \beta_j^2 r^2) H_n^{(1)}(\beta_j r) + \beta_j r H_{n-1}^{(1)}(\beta_j r) \quad (32) $$

$$ j = 1, 2 $$

$$ {}_2\mathfrak{D}^{(i)}_{nr} = -(n^2 + n)\, H^{(2)}_n (\beta_2 r) + n\beta_2 r\, H^{(2)}_{n-1}(\beta_j r) $$

$$ {}_2\mathcal{E}^{(i)}_{nr} = -(n^2 + n - \frac{1}{2}\beta^2_2 r^2)\, H^{(2)}_n(\beta_2 r) + \beta_2 r\, H^{(2)}_{n-1}(\beta_2 r) \tag{33} $$

and

$$ H'_n(x) = \frac{dH_n}{dx} $$

$$ J'_n(x) = \frac{dJ_n}{dx} \tag{34} $$

Equations (19) through (28) are the general expression for the stresses and displacements in regions Nos. 1 and 2. To evaluate the coefficient A_n, etc., use the conditions of continuity at $r = b$ and the boundary conditions at $r = a$.

At $r = b$

$$ \tau_{rr_{(2)}} = \tau_{rr_{(1)}} $$

$$ M_{n\,2}D^{(i)}_{nb} + N_{n\,2}\mathfrak{D}^{(i)}_{nb} + R_{n\,2}D^{(R)}_{nb} + S_{n\,2}\mathfrak{D}^{(R)}_{nb} - \nu(A_{n\,1}D^{(R)}_{nb} + B_{n\,1}\mathfrak{D}^{(R)}_{nb}) = \nu\, \phi_0 \epsilon_n i^n\, {}_1D^{(i)}_{nb} \tag{35} $$

$$ \tau_{r\theta_{(2)}} = \tau_{r\theta_{(1)}} $$

$$ M_{n\,2}E^{(i)}_{nb} + N_{n\,2}\mathcal{E}^{(i)}_{nb} + R_{n\,2}E^{(R)}_{nb} + S_{n\,2}\mathcal{E}^{(R)}_{nb} - \nu(A_{n\,1}E^{(R)}_{nb} + B_{n\,1}\mathcal{E}^{(R)}_{nb}) = \nu\, \phi_0 \epsilon_n i^n\, {}_1E^{(i)}_{nb} \tag{36} $$

$$u_{r_{(2)}} = u_{r_{(1)}}$$

$$M_n \alpha_2 b H_n^{(2)'}(\alpha_2 b) + N_n n H_n^{(2)}(\beta_2 b) + R_n \alpha_2 b H_n^{(1)'}(\alpha_2 b) + S_n n H_n^{(1)}(\beta_2 b)$$

$$- \left[A_n \alpha_1 b H_n^{(1)'}(\alpha_1 b) + B_n n H_n^{(1)}(\beta_1 b) \right] = \phi_0 \epsilon_n i^n \alpha_1 b J_n'(\alpha_1 b) \qquad (37)$$

$$u_{\theta_{(2)}} = u_{\theta_{(1)}}$$

$$M_n n H_n^{(2)}(\alpha_2 b) + N_n \beta_2 b H_n^{(2)'}(\beta_2 b) + R_n n H_n^{(1)}(\alpha_2 b) + S_n \beta_2 b H_n^{(1)'}(\beta_2 b)$$

$$- \left[A_n n H_n^{(1)}(\alpha_1 b) + B_n \beta_1 b H_n^{(1)'}(\beta_1 b) \right] = \phi_0 \epsilon_n i^n n J_n(\alpha_1 b) \qquad (38)$$

At $r = a$

$$\tau_{rr_{(2)}} = 0$$

$$M_n^{(2)} {}_2D_{na}^{(i)} + N_{n\,2} \mathfrak{D}_{na}^{(i)} + R_{n\,2} D_{na}^{(R)} + S_{n\,2} \mathfrak{D}_{na}^{(R)} = 0 \qquad (39)$$

$$\tau_{r\theta_{(2)}} = 0$$

$$M_{n\,2} E_{na}^{(i)} + N_{n\,2} \mathcal{E}_{na}^{(i)} + R_{n\,2} E_{na}^{(R)} + S_{n\,2} \mathcal{E}_{na}^{(R)} = 0 \qquad (40)$$

where $\nu = \mu_1/\mu_2$ is the ratio of the shear moduli of regions Nos. 1 and 2.

Thus, the coefficients A_n, B_n, M_n, N_n, R_n, and S_n are determined by the six simultaneous Equations (35) through (40).

Using matrix notation, Equations (35) through (40) can be conveniently expressed as follows

$$\left[a_{ij}\right]\left\{c_j\right\}=\left\{b_i\right\} \tag{41}$$

where

$$\left[a_{ij}\right]=\begin{bmatrix} \nu_1 D_{nb}^{(R)} & \nu_1 \mathcal{D}_{nb}^{(R)} & {}_2D_{nb}^{(i)} & {}_2\mathcal{D}_{nb}^{(i)} & {}_2D_{nb}^{(R)} & {}_2\mathcal{D}_{nb}^{(R)} \\ \nu_1 E_{nb}^{(R)} & \nu_1 \mathcal{E}_{nb}^{(R)} & {}_2E_{nb}^{(i)} & {}_2\mathcal{E}_{nb}^{(i)} & {}_2E_{nb}^{(R)} & {}_2\mathcal{E}_{nb}^{(R)} \\ \alpha_1 bH_n^{(1)'}(\alpha_1 b) & nH_n^{(1)}(\beta_1 b) & \alpha_2 bH_n^{(2)'}(\alpha_2 b) & nH_n^{(2)}(\beta_2 b) & \alpha_2 bH_n^{(1)'}(\alpha_2 b) & nH_n^{(1)}(\beta_2 b) \\ nH_n^{(1)}(\alpha_1 b) & \beta_1 bH_n^{(1)'}(\beta_1 b) & nH_n^{(2)}(\alpha_2 b) & \beta_2 bH_n^{(2)'}(\beta_2 b) & nH_n^{(1)}(\alpha_2 b) & \beta_2 bH_n^{(1)'}(\beta_2 b) \\ 0 & 0 & {}_2D_{na}^{(i)} & {}_2\mathcal{D}_{na}^{(i)} & {}_2D_{na}^{(R)} & {}_2\mathcal{D}_{na}^{(R)} \\ 0 & 0 & {}_2E_{na}^{(i)} & {}_2\mathcal{E}_{na}^{(i)} & {}_2E_{na}^{(R)} & {}_2\mathcal{E}_{na}^{(R)} \end{bmatrix} \tag{42}$$

$$\left\{ c_j \right\} = \left\{ \begin{array}{c} -A_n \\ -B_n \\ M_n \\ N_n \\ R_n \\ S_n \end{array} \right\} \tag{43}$$

and

$$\left\{ b_i \right\} = \phi_0 \epsilon_n i^n \left\{ \begin{array}{c} \nu_1 D_{nb}^{(i)} \\ \nu_1 E_{nb}^{(i)} \\ \alpha_1 b J_n'(\alpha_1 b) \\ n J_n(\alpha_1 b) \\ 0 \\ 0 \end{array} \right\} \tag{44}$$

Hence

$$\left\{ c_i \right\} = \frac{\left[A_{ji} \right]}{\left| a \right|} \left\{ b_j \right\} \tag{45}$$

where $\left[A_{ji} \right]$ is the adjoint of $\left[a_{ij} \right]$ and $|a|$ is the determinant of $\left\{ a_{ij} \right\}$.

REDUCTION TO THE SIMPLE CAVITY CASE

Consider the case when the material constants of the two regions are the same:

$$\lambda_1 = \lambda_2 = \lambda$$

$$\mu_1 = \mu_2 = \mu$$

$$\rho_1 = \rho_2 = \rho$$

Therefore

$$\alpha_1 = \alpha_2 = \alpha$$

$$\beta_1 = \beta_2 = \beta$$

$$\nu = 1$$

It follows that

$$
\begin{aligned}
{}_1D^{(R)}_{nr} &= {}_2D^{(R)}_{nr} \\
{}_1\mathfrak{D}^{(R)}_{nr} &= {}_2\mathfrak{D}^{(R)}_{nr} \\
{}_1E^{(R)}_{nr} &= {}_2E^{(R)}_{nr} \\
{}_1\mathcal{E}^{(R)}_{nr} &= {}_2\mathcal{E}^{(R)}_{nr} \qquad (46)
\end{aligned}
$$

Furthermore, using the relationship between $H_n^{(1)}$ and $H_n^{(2)}$

$$H_n^{(1)}(x) + H_n^{(2)}(x) = 2\,J_n(x) \tag{47}$$

it can be shown that

$${}_2D_{nr}^{(i)} + {}_2D_{nr}^{(R)} = 2\;{}_1D_{nr}^{(i)} \tag{48}$$

$${}_2E_{nr}^{(i)} + {}_2E_{nr}^{(R)} = 2\;{}_1E_{nr}^{(i)} \tag{49}$$

Substituting Equations (46), (48), and (49) into Equation (45), the undetermined coefficients for the two potentials in region No. 1 are

$$A_n = -\phi_0 \epsilon_n i^n \frac{\begin{vmatrix} {}_1D_{na}^{(i)} & {}_1\mathfrak{D}_{na}^{(R)} \\ {}_1E_{na}^{(i)} & {}_1\mathfrak{E}_{na}^{(R)} \end{vmatrix}}{\begin{vmatrix} {}_1D_{na}^{(R)} & {}_1\mathfrak{D}_{na}^{(R)} \\ {}_1E_{na}^{(R)} & {}_1\mathfrak{E}_{na}^{(R)} \end{vmatrix}} \tag{50}$$

and

$$B_n = -\phi_0 \epsilon_n i^n \frac{\begin{vmatrix} {}_1D_{na}^{(R)} & {}_1D_{na}^{(i)} \\ {}_1E_{na}^{(R)} & {}_1E_{na}^{(i)} \end{vmatrix}}{\begin{vmatrix} {}_1D_{na}^{(R)} & {}_1\mathfrak{D}_{na}^{(R)} \\ {}_1E_{na}^{(R)} & {}_1\mathcal{E}_{na}^{(R)} \end{vmatrix}} \tag{51}$$

These expressions for A_n and B_n are the same as those in reference 1. The remaining coefficients M_n, N_n, R_n, and S_n are

$$\begin{aligned} M_n &= \frac{1}{2}\phi_0 \epsilon_n i^n \\ N_n &= 0 \\ S_n &= B_n \\ R_n &= \frac{1}{2}\phi_0 \epsilon_n i^n + A_n \end{aligned} \tag{52}$$

Thus the shear potential in region No. 2 is obviously the same as that in region No. 1. The total scalar potential in region No. 2 is

$$\phi_{(2)} = \phi^{(i)}_{(2)} + \phi^{(R)}_{(2)} = \sum_{n=0}^{\infty} \frac{1}{2} \phi_0 \epsilon_n i^n H_n^{(2)} (\alpha r) \cos n\theta +$$

$$(\frac{1}{2} \phi_0 \epsilon_n i^n + A_n) H_n^{(1)} (\alpha r) \cos n\theta$$

$$= \sum_{n=0}^{\infty} \left[\phi_0 \epsilon_n i^n J_n(\alpha r) + A_n H_n^{(1)} (\alpha r) \right] \cos n\theta \tag{53}$$

which is identical to the scalar potential in region No. 1. Therefore, the problem is reduced to a simple cavity case.

SECTION V

NUMERICAL RESULTS AND DISCUSSIONS

Numerical results are obtained for $\tau_{rr_{(1)}}$ and $\tau_{\theta\theta_{(1)}}$ at r = b and $\tau_{\theta\theta_{(2)}}$ at r = a. This is accomplished by summing the series in Equations (23), (24), and (27), respectively.

For reasons of expediency, the expressions for the stresses are nondimensionalized. The important nondimensional parameters can then be properly identified, and the effects of these parameters on the stresses in the cylinder as well as in the surrounding medium can be evaluated.

To nondimensionalize these expressions, recall the expression for the incident wave given in Equation (7) as

$$\phi_{(1)}^{(i)} = \phi_0 e^{i(\alpha_1 x - \omega t)}$$

Therefore, the stress in the incident wave in the direction of propagation is

$$\tau_{xx} = -(\lambda_1 + 2\mu_1)\alpha_1^2 \phi_0 e^{i(\alpha_1 x - \omega t)} \tag{54}$$

It follows that $-(\lambda_1 + 2\mu_1)\alpha_1^2 \phi_0$ is the stress amplitude of the incident wave. Denote it as

$$\tau_0 = -(\lambda_1 + 2\mu_1)\alpha_1^2 \phi_0 = -\beta_1^2 \mu_1 \phi_0 \tag{55}$$

Thus, Equations (23), (24), and (27) can be nondimensionalized by dividing through by τ_0 as shown below:

$$\tau^*_{rr_{(1)}} = \frac{\tau_{rr_{(1)}}}{\tau_0} = - \frac{2}{\phi_0 \beta_1^2 r^2} \sum_{n=0}^{\infty} (\phi_0 \epsilon_n i^n {}_1D^{(i)}_{nr} + A_{n\,1}D^{(R)}_{nr}$$

$$+ B_{n\,1}\mathfrak{D}^{(R)}_{nr}) \cos n\theta \qquad (56)$$

$$\tau^*_{\theta\theta_{(1)}} = \frac{\tau_{\theta\theta_{(1)}}}{\tau_0} = - \frac{2}{\phi_0 \beta_1^2 r^2} \sum_{n=0}^{\infty} (\phi_0 \epsilon_n i^n {}_1F^{(i)}_{nr} + A_{n\,1}F^{(R)}_{nr}$$

$$- B_{n\,1}\mathfrak{D}^{(R)}_{nr}) \cos n\theta \qquad (57)$$

$$\tau^*_{\theta\theta_{(2)}} = \frac{\tau_{\theta\theta_{(2)}}}{\tau_0} = - \frac{2}{\phi_0 \nu \beta_1^2 r^2} \sum_{n=0}^{\infty} (M_{n\,2}F^{(i)}_{nr} - N_{n\,2}\mathfrak{D}^{(i)}_{nr}$$

$$+ R_{n\,2}F^{(R)}_{nr} - S_{n\,2}\mathfrak{D}^{(R)}_{nr}) \cos n\theta \qquad (58)$$

where $\nu = \mu_1/\mu_2$ is the ratio of shear moduli of regions Nos. 1 and 2.

By letting r = b in Equations (56) and (57), and r = a in Equation (58) we have, for example,

$$\tau^*_{rr_{(1)}\atop r=b} = -\frac{2}{\phi_0 \beta_1^2 b^2} \sum_{n=0}^{\infty} \left(\phi_0 \epsilon_n i^n \, {}_1D_{nb}^{(i)} + A_{n\,1} D_{nb}^{(R)} + B_{n\,1} \mathfrak{D}_{nb}^{(R)}\right) \cos n\theta \tag{59}$$

with similar expressions for Equations (57) and (58).

By examining the arguments in ${}_1D_{nb}^{(i)}$ ... etc., that are associated with various terms in Equation (59) etc., it is found that five essential nondimensional parameters exist. These are

$$\nu = \mu_1/\mu_2$$

$$\gamma = \frac{\alpha_2}{\alpha_1} = \frac{c_{\alpha_1}}{c_{\alpha_2}}$$

ratio of the dilatational phase velocities of regions Nos. 1 and 2

$$k_1^2 = \left(\frac{\beta_1}{\alpha_1}\right)^2 = \left(\frac{c_{\alpha_1}}{c_{\beta_1}}\right)^2 = \frac{2(1-\sigma_1)}{1-2\sigma_1}$$

$$k_2^2 = \left(\frac{\beta_2}{\alpha_2}\right)^2 = \left(\frac{c_{\alpha_2}}{c_{\beta_2}}\right)^2 = \frac{2(1-\sigma_2)}{1-2\sigma_2}$$

ratio of dilatational phase velocities to distortional phase velocities in regions Nos. 1 and 2

$$\eta = b/a$$

ratio of outer radius to inner radius of the cylinder

By substituting these expressions into Equation (59) ... etc., the final expressions for $\tau^*_{rr_{(1)}}$ $\tau^*_{\theta\theta_{(1)}}$ at r = b and $\tau^*_{\theta\theta_{(2)}}$ at r = a are obtained

$$\tau^*_{\substack{rr_{(1)} \\ r=b}} = - \frac{2}{\phi_0 k_1^2 \eta^2 (\alpha_1 a)^2} \sum_{n=0}^{\infty} \left(\phi_0 \epsilon_n i^n \, {}_1D^{(i)}_{nb} + A_n \, {}_1D^{(R)}_{nb} + B_n \, {}_1\mathfrak{D}^{(R)}_{nb}\right) \cos n\theta \tag{60}$$

$$\tau^*_{\substack{\theta\theta_{(1)} \\ r=b}} = - \frac{2}{\phi_0 k_1^2 \eta^2 (\alpha_1 a)^2} \sum_{n=0}^{\infty} \left(\phi_0 \epsilon_n i^n \, {}_1F^{(i)}_{nb} + A_n \, {}_1F^{(R)}_{nb} - B_n \, {}_1\mathfrak{D}^{(R)}_{nb}\right) \cos n\theta \tag{61}$$

$$\tau^*_{\substack{\theta\theta_{(2)} \\ r=a}} = - \frac{2}{\phi_0 \nu k_1^2 (\alpha_1 a)^2} \sum_{n=0}^{\infty} \left(M_n \, {}_2F^{(i)}_{na} - N_n \, {}_2\mathfrak{D}^{(i)}_{na} + R_n \, {}_2F^{(R)}_{na} - S_n \, {}_2\mathfrak{D}^{(R)}_{na}\right) \cos n\theta \tag{62}$$

The expressions for ${}_1D^{(i)}_{nb} \ldots {}_2\mathfrak{D}^{(R)}_{na}$ can be expressed in terms of the incident dimensionless wave number $(\alpha_1 a)$ and the parameters defined above. For example

$$\begin{aligned} {}_1D^{(i)}_{nb} &= \left(n^2 + n - \frac{1}{2} \beta_1^2 b^2\right) J_n(\alpha_1 b) - \alpha_1 b \, J_{n-1}(\alpha_1 b) \\ &= \left[n^2 + n - \frac{1}{2} k_1^2 \eta^2 (\alpha_1 a)^2\right] J_n(\eta\alpha_1 a) - \eta\alpha_1 a J_{n-1}(\eta\alpha_1 a) \end{aligned} \tag{63}$$

It is apparent that the stresses in the cylinder as well as in the surrounding elastic medium are not only a function of the incident wave frequency, but also depend on the four physical parameters of the two media and a geometrical parameter of the cylinder.

By restoring $e^{-i\omega t}$, Equations (60) through (62) have the form

$$(R + i I) e^{-i\omega t} = (R^2 + I^2)^{1/2} e^{-i(\omega t - \delta)} \tag{64}$$

The real part represents the stresses at $t = 0$ and the imaginary part gives the stresses at $t = T/4$, where $T = 2\pi/\omega$ is the period of the incident wave. The absolute values $(R^2 + I^2)^{1/2}$ correspond to the maximum values of $\tau^*_{rr_{(1)}}$, $\tau^*_{\theta\theta_{(1)}}$ and $\tau^*_{\theta\theta_{(2)}}$ which occur in the interval $t = 0$ to $t = T/4$. The phase angles are given by $\delta = \arctan I/R$.

Numerical results are obtained for two cases corresponding to a soft and a stiff cylindrical lining.

SOFT CYLINDRICAL LINING

In this case the dimensionless physical parameters are assumed to be

$\nu = 2.90$

$\gamma = 1.50$

$\sigma_1 = .25$

$\sigma_2 = .20$

$\eta = 1.05,\ 1.1$ and 1.20

The results are shown in Figs. 2 through 6.

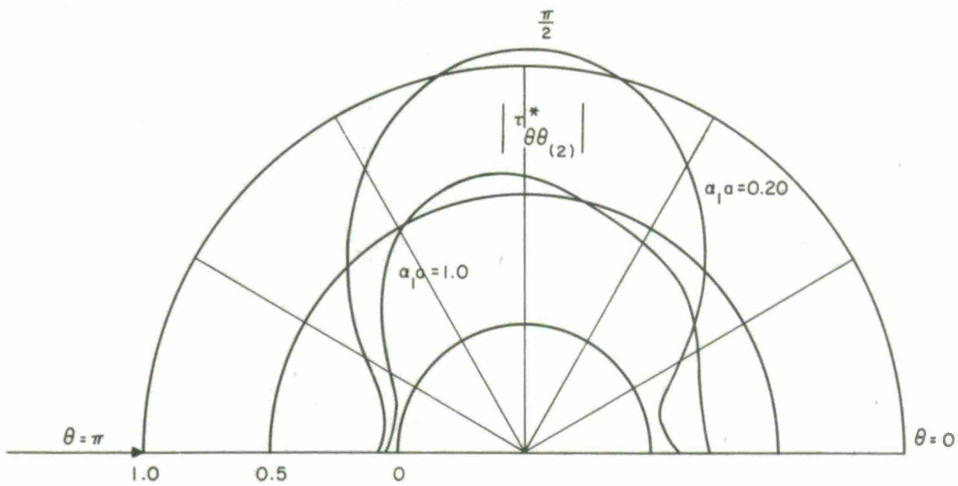

Fig. 2 Distribution of Normalized Tangential Stress at $r = a$ and $\eta = 1.1$ (Case I)

Fig. 2 shows the angular distribution of $\left|\tau^*_{\theta\theta_{(2)}}\right|$ for $\eta = 1.1$ at $r = a$ for two wave numbers: $\alpha_1 a = .20$ and $\alpha_1 a = 1.0$. At the low wave number, the distribution is nearly symmetrical with respect to the y axis; at $\alpha_1 a = 1.0$, the peak stress is shifted toward the incident side of the cylinder. This is also found in reference 2.

Fig. 3 shows the stresses as a function of $\alpha_1 a$ and η at $t = 0$ and $t = T/4$. For this case, an increase in η does not change the stress appreciably; in fact, as η increases, the stress also increases.

Fig. 4 shows the stresses at $\theta = \pi/2$, π and $r = a$ for $.10 \leq \alpha_1 a \leq 2.0$. The maximum stresses at $\pi/2$ occur at $\alpha_1 a \approx .25$; this agrees with the results of reference 1.

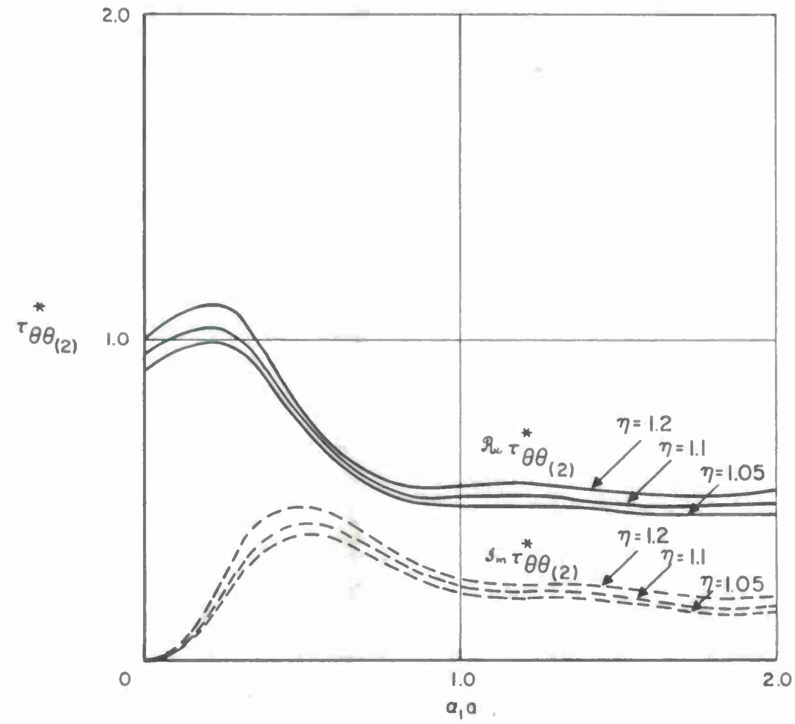

Fig. 3 Normalized Real and Imaginary Tangential Stresses at $r = a, \quad \theta = \dfrac{\pi}{2}$ for Various η (Case I)

Figs. 5 and 6 show the effects of η on the stresses in medium (1) at $r = b$. It is seen that $|\tau^*_{\theta\theta_{(1)}}|$ at $\pi/2$ decreases as η increases, with the peak value again occurring at $\alpha_1 a \approx .25$. On the other hand, $|\tau^*_{rr_{(1)}}|$ in all cases increases as η increases. This is apparent since the rigidity of the cylinder is increased as η increases; hence, higher radial stresses are produced at the boundary.

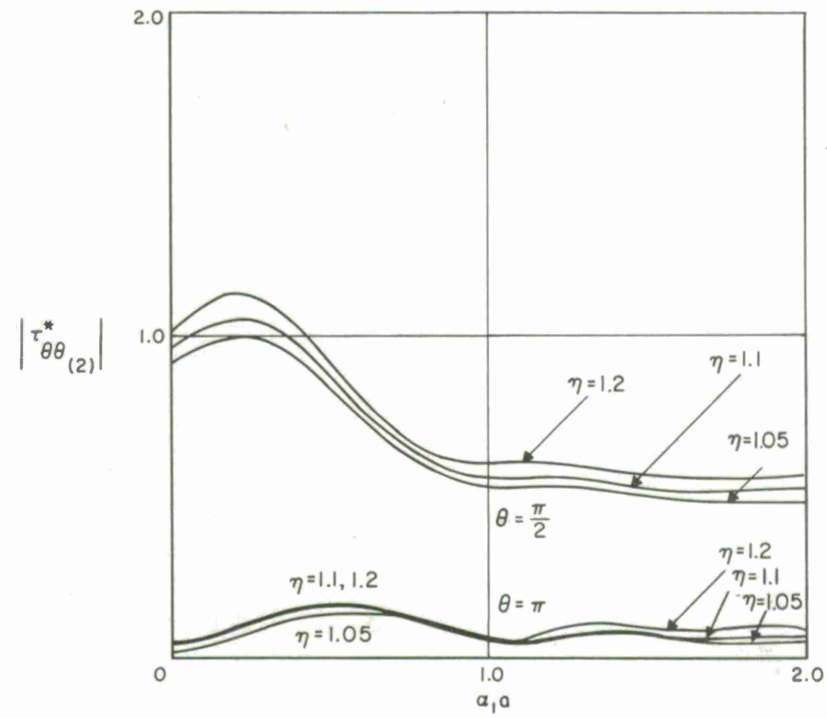

Fig. 4 Normalized Tangential Stress at r = a, $\theta = \frac{\pi}{2}, \pi$ for Various η (Case I)

STIFF CYLINDRICAL LINING

The dimensionless parameters are assumed to be

$\nu = .31$

$\gamma = .70$

$\sigma_1 = .25$

$\sigma_2 = .30$

$\eta = 1.05,\ 1.1$ and 1.2

The results for this case (Case II) are shown in Figs. 7 through 11.

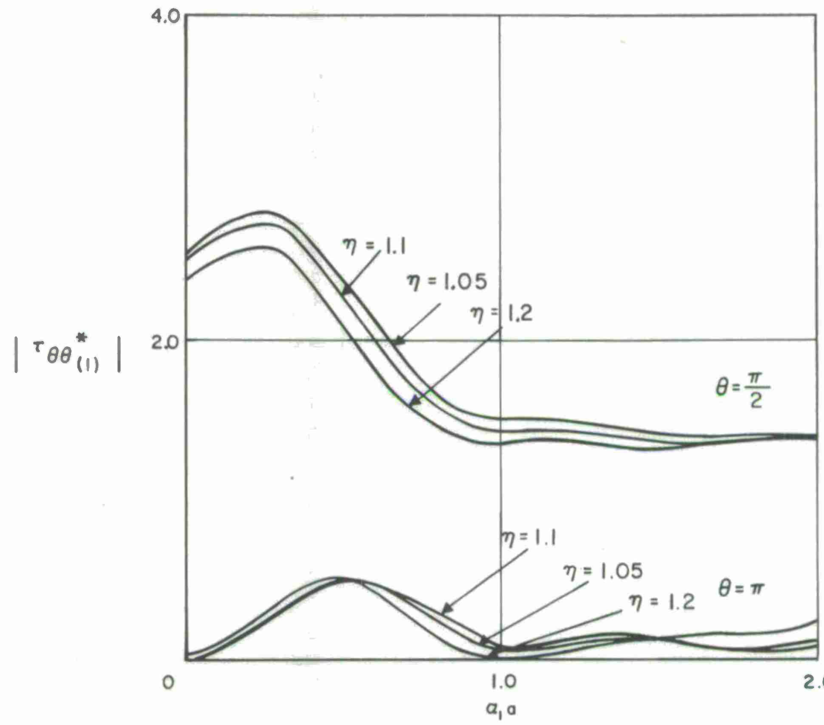

Fig. 5 Normalized Tangential Stress at $r = b$, $\theta = \frac{\pi}{2}, \pi$ for Various η (Case I)

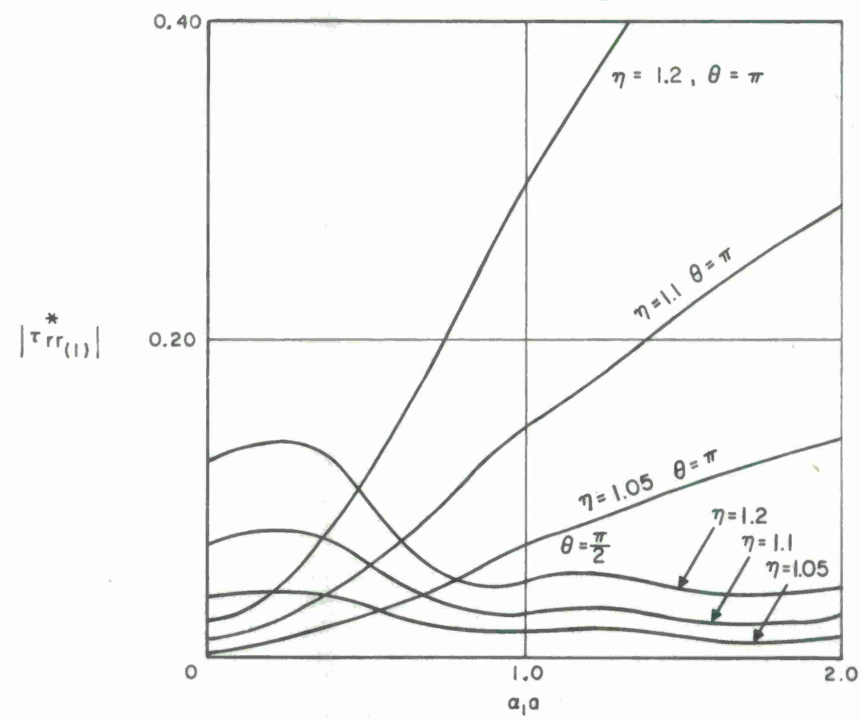

Fig. 6 Normalized Radial Stress at $r = b$, $\theta = \frac{\pi}{2}, \pi$ for Various η (Case I)

Fig. 7 shows the angular distribution of $|\tau^*_{\theta\theta_{(2)}}|$ for $\eta = 1.1$ at $r = a$ for two wave numbers; $\alpha_1 a = .20$ and $\alpha_1 a = 1.0$. Note that the shifting exhibited in the case of the soft cylindrical lining (Case I) also occurs in this case. The magnitude of the stresses are, however, much higher.

Fig. 9 shows the effects of η on stresses in the cylinder. In this case, the stress decreases as η increases; this is in contrast to the previous soft cylinder case.

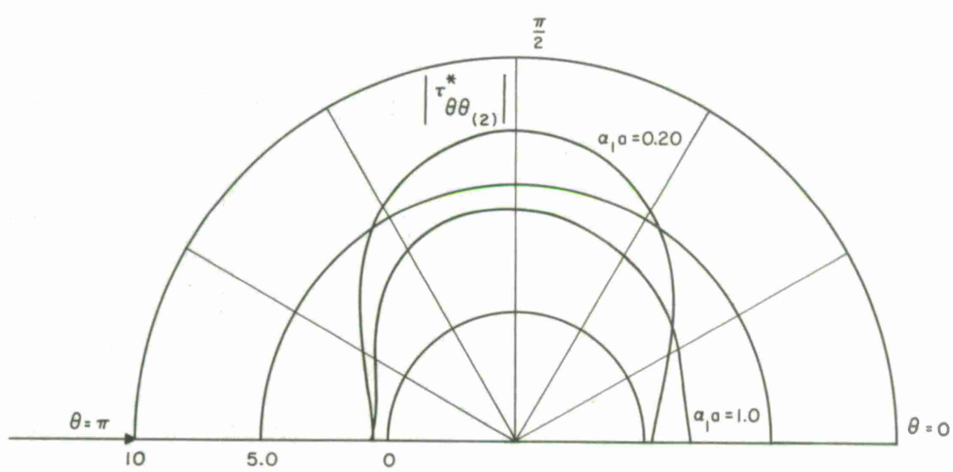

Fig. 7 Distribution of Normalized Tangential Stress at $r = a$ and $\eta = 1.1$ (Case II)

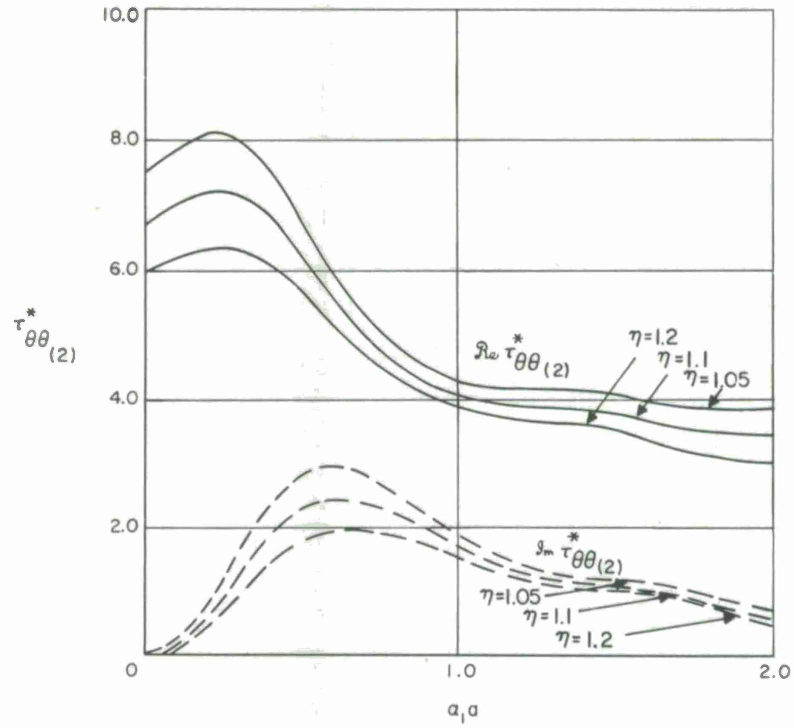

Fig. 8 Normalized Real and Imaginary Tangential Stresses at $r = a$, $\theta = \frac{\pi}{2}$, for Various η (Case II)

Figs. 10 and 11 show the effects of η on the stress in region No. 1. The general trend is the same as in (Case I).

The contrast in the results for the stiff lining and the soft lining should be emphasized. Both cases exhibit the tendency for stress to concentrate at large wave numbers on the incident side of the cylinder; however, the tangential stress in the lining is significantly higher in the case of the stiff liner. The wave numbers for maximum stress at $\theta = \pi/2$ and π are approximately the same for both the stiff and the soft lining, but the effects of increased lining thickness on tangential stress are opposite in the two cases. The stress decreases with thickness for the stiff liner, but increases with thickness in the case of the soft liner.

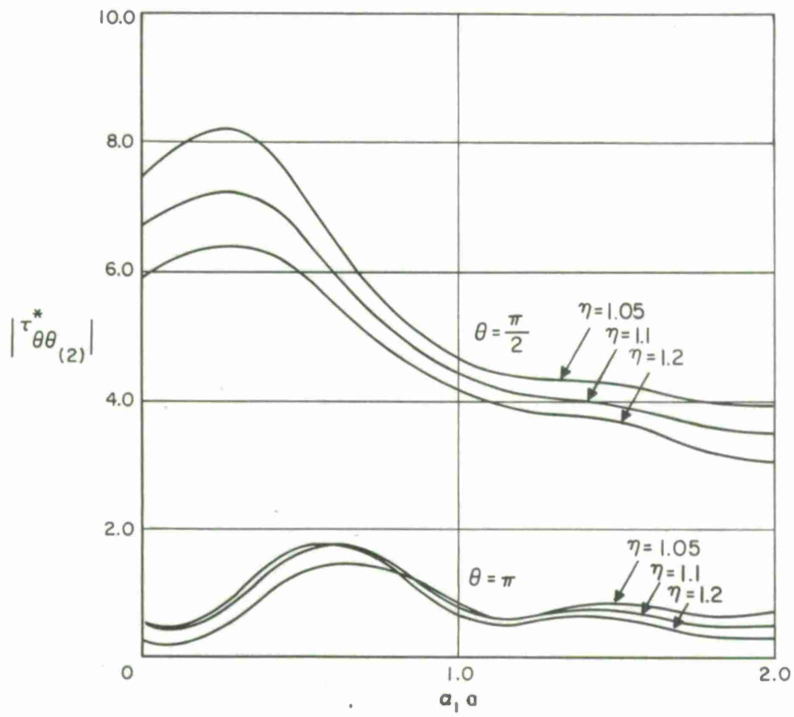

Fig. 9 Maximum Normalized Tangential Stress at $r = a$, $\theta = \frac{\pi}{2}$, π for Various η (Case II)

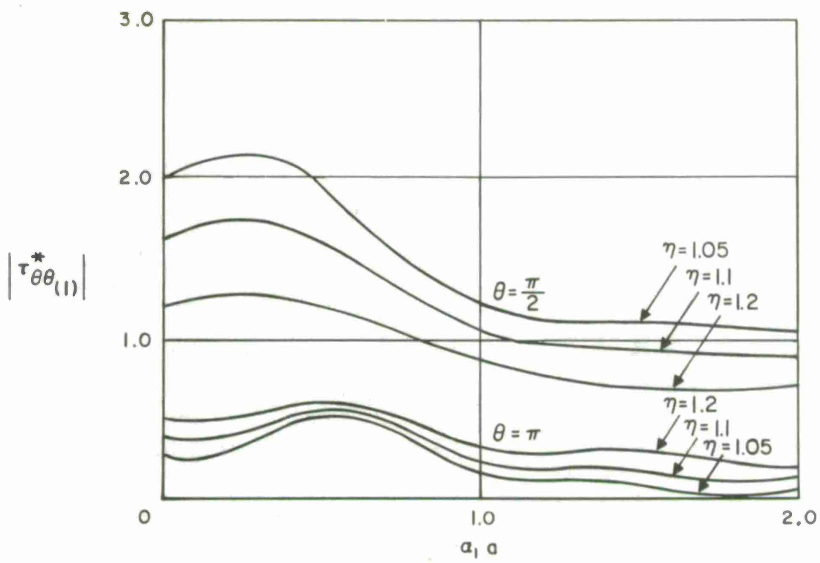

Fig. 10 Normalized Tangential Stress in Medium (1) at $r = b, \theta = \frac{\pi}{2}, \pi$ for Various η (Case II)

Other points of contrast are:

(a) The variance of tangential stress in the cylinder with thickness is less in the case of the soft liner.

(b) The tangential stress in the infinite medium at the boundary of the lining is less in the case of the stiff lining. In addition, increasing thickness markedly reduces this stress, significantly more so than for the soft lining.

(c) The radial stresses in the infinite medium at the boundary of the lining are much less in the case of the soft lining. In both cases, this stress reduces rapidly with decreasing thickness.

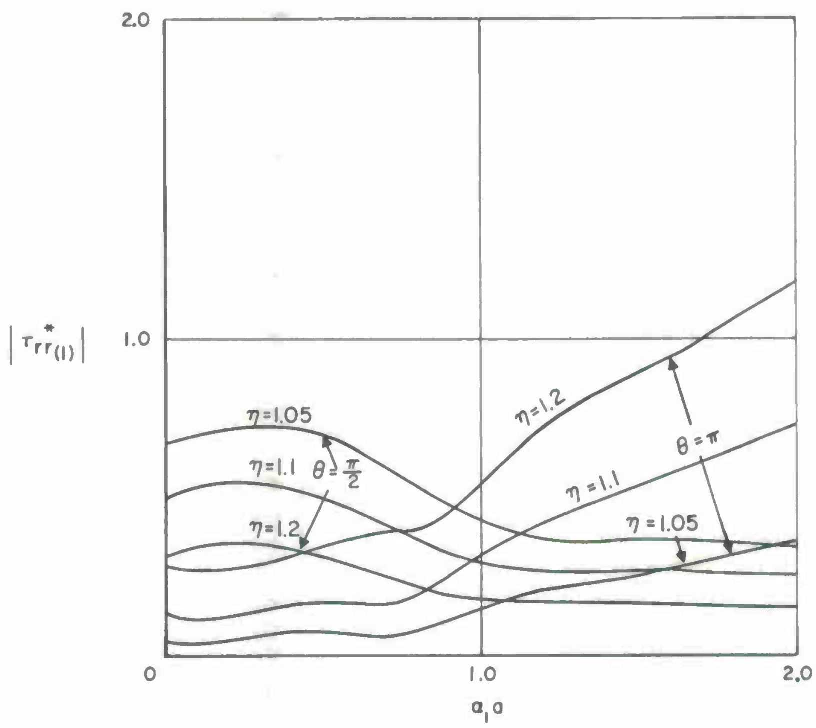

Fig. 11 Normalized Radial Stress in Medium (1) at $r = b, \theta = \frac{\pi}{2}, \pi$ for Various η (Case II)

APPENDIX

THE EXACT TWO-DIMENSIONAL SOLUTION FOR AN ELASTIC CYLINDRICAL LINING IN AN INFINITE ELASTIC MEDIUM UNDER BIAXIAL COMPRESSIVE LOADING

This appendix derives expressions for the stresses in an elastic cylindrical lining in an infinite elastic medium under biaxial compressive forces. This static solution represents the infinite wave length (zero frequency) solution to which the traveling wave solution must converge in the limit. This is accomplished by using a superposition approach in which the compatibility equations for the stress and displacement at the interface between the lining and the infinite medium are expressed in terms of unknown radial and tangential tractions assumed to act on the lining and the boundary of the infinite medium. The solution derived is a plane strain solution.

STATIC STRESS AND DISPLACEMENT OF AN UNLINED CYLINDRICAL CAVITY BOUNDARY DUE TO THE BIAXIAL COMPRESSIVE FIELD $\tau_{xx} = -\tau_0,\ \tau_{yy} = \epsilon\tau_0$

Consider an infinite elastic medium with a cylindrical cavity of radius "b" under biaxial compression $\tau_{xx} = -\tau_0$, $\tau_{yy} = \epsilon\tau_0$ where $\epsilon = -\sigma_1/1-\sigma_1$ (Fig. 12). The subscript 1 is used to identify the infinite medium as opposed to the lining considered below. The displacements at the cavity boundary due to the biaxial compressive field[7] are

$$u'_{r(1)}\Big|_{r=b} = -\frac{\tau_0 b}{E_1}(1+\sigma_1)\left[1 + 2(1-2\sigma_1)\cos 2\theta\right] \tag{65}$$

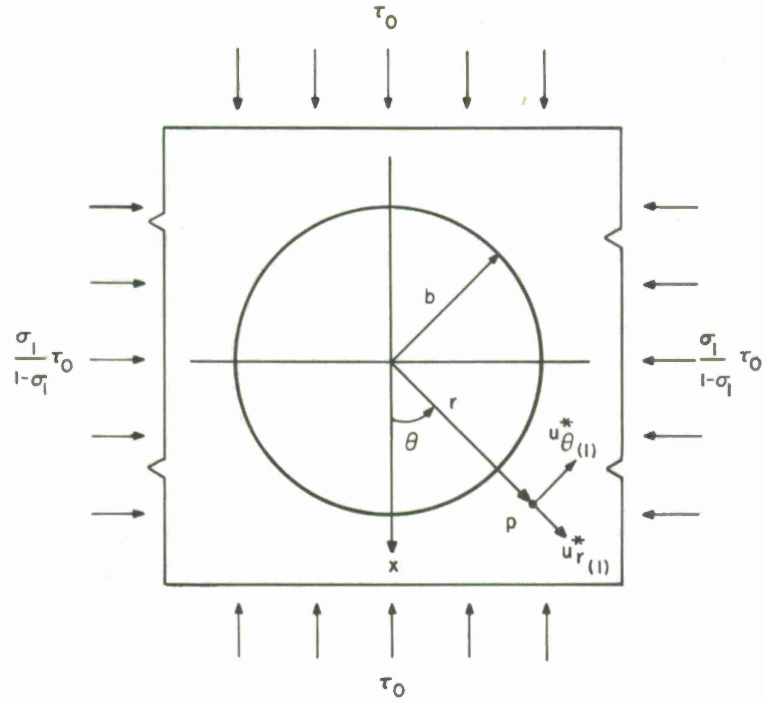

Fig. 12 Unlined Cavity

$$u'_{\theta_{(1)}\,r=b} = \frac{2\tau_0 b}{E_1}\left[(1 - 2\sigma_1)(1 + \sigma_1)\sin 2\theta\right] \tag{66}$$

Reference 8 gives the tangential stress at the boundary of the cavity due to the biaxial compressive field; i.e.,

$$\tau'_{\theta\theta_{(1)}\,r=b} = \frac{\tau_0}{1-\sigma_1}\left[-1 + (2 - 4\sigma_1)\cos 2\theta\right] \tag{67}$$

The radial and shear stresses vanish at the boundary.

STATIC STRESS AND DISPLACEMENT OF AN UNLINED CAVITY BOUNDARY DUE TO APPLIED BOUNDARY TRACTIONS $\bar{\tau}_{rr_{(1)}}$ AND $\bar{\tau}_{r\theta_{(1)}}$

Consider an infinite elastic medium with a cylindrical cavity of radius "b" under action of applied radial and tangential boundary tractions $\bar{\tau}_{rr_{(1)}}$ and $\bar{\tau}_{r\theta_{(1)}}$, respectively. Let

$$\bar{\tau}_{rr_{(1)}} = \bar{\tau}_{rr_0} + \bar{\tau}_{rr_1} \cos 2\theta$$

$$\bar{\tau}_{r\theta_{(1)}} = \bar{\tau}_{r\theta_1} \sin 2\theta \tag{68}$$

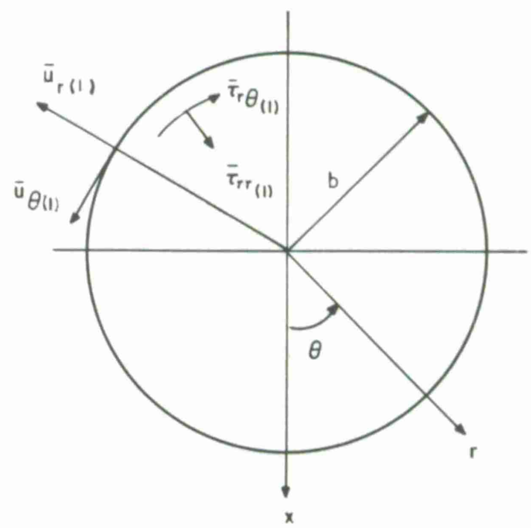

Fig. 13 Boundary Tractions - Unlined Cavity

where $\bar{\tau}_{rr_0}$, $\bar{\tau}_{rr_1}$, and $\bar{\tau}_{r\theta_1}$ are arbitrary constants. Fig. 13 illustrates the problem. At the boundary (r = b) displacements and stresses are

$$\bar{u}_{r_{(1)}\atop r=b} = -\frac{\bar{\tau}_{rr_0}}{E_1}(1+\sigma_1) + \frac{2(1+\sigma_1)b}{E_1}\left[-\bar{\tau}_{rr_1}\left(\frac{5}{6}-\sigma_1\right) + \bar{\tau}_{r\theta_1}\left(\frac{2}{3}-\sigma_1\right)\right]\cos 2\theta \tag{69}$$

$$\bar{u}_{\theta_{(1)}\atop r=b} = -\frac{(1+\sigma_1)b}{E_1}\left[-\bar{\tau}_{rr_1}\left(\frac{4}{3}-2_{\sigma_1}\right) + \bar{\tau}_{r\theta_1}\left(\frac{5}{3}-2\nu\right)\right]\sin 2\theta \tag{70}$$

and

$$\bar{\tau}_{rr_{(1)}\atop r=b} = \bar{\tau}_{rr_0} + \bar{\tau}_{rr_1}\cos 2\theta$$

$$\bar{\tau}_{\theta\theta_{(1)}\atop r=b} = -\bar{\tau}_{rr_0} + (\bar{\tau}_{rr_1} - 2\,\bar{\tau}_{r\theta_1})\cos 2\theta$$

$$\bar{\tau}_{r\theta_{(1)}\atop r=b} = \bar{\tau}_{r\theta_1}\sin 2\theta \tag{71}$$

STATIC DISPLACEMENTS AND STRESSES IN AN ELASTIC CYLINDRICAL LINER UNDER ARBITRARY BOUNDARY TRACTIONS

The generalized stress function and stresses for plane strain in cylindrical coordinates (Fig. 14) are

$$\psi_{(2)}(r, \theta) = C_1 r^2 \ell n r + C_2 r^2 + C_3 \ell n r + C_4 + \left(C_5 r^2 + C_6 r^4 + \frac{C_7}{r^2} + C_8 \right) \cos 2\theta \tag{72}$$

$$\tau_{rr_{(2)}} = C_1 (1 + 2\ell n r) + 2C_2 + \frac{C_3}{r^2} - \left(2C_5 + \frac{6C_7}{r^4} + \frac{4C_8}{r^2} \right) \cos 2\theta \tag{73}$$

$$\tau_{\theta\theta_{(2)}} = C_1 (3 + 2\ell n r) + 2C_2 - \frac{C_3}{r^2} + \left(2C_5 + 12C_6 r^2 + \frac{6C_7}{r^4} \right) \cos 2\theta \tag{74}$$

$$\tau_{r\theta_{(2)}} = \left(2C_5 + 6C_6 r^2 - \frac{6C_7}{r^4} - \frac{2C_8}{r^2} \right) \sin 2\theta \tag{75}$$

where the subscript 2 identifies the elastic lining as opposed to the elastic infinite medium.

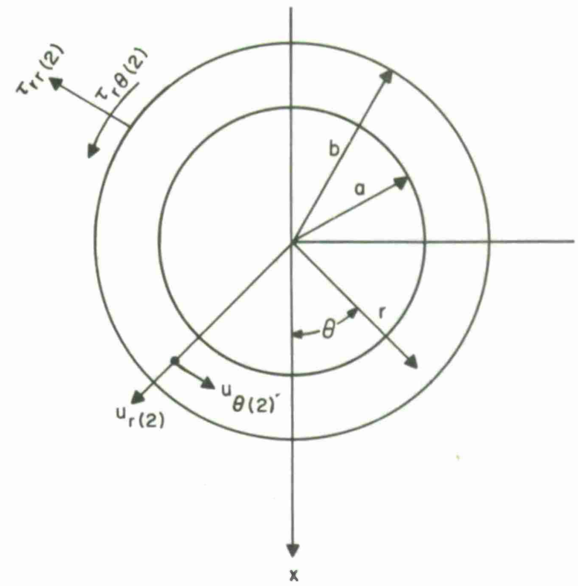

Fig. 14 Elastic Cylinder with Boundary Tractions

The stress-displacement relationships are

$$\epsilon_{rr_{(2)}} = \frac{\partial u_{r_{(2)}}}{\partial r} = \frac{1+\sigma_2}{E_2}\left[(1-\sigma_2)\,\tau_{rr_{(2)}} - \sigma_2\,\tau_{\theta\theta_{(2)}}\right]$$

$$\epsilon_{\theta\theta_{(2)}} = \frac{u_{r_{(2)}}}{r} + \frac{1}{r}\,\frac{\partial u_{\theta_{(2)}}}{\partial \theta} = \frac{1+\sigma_2}{E_2}\left[(1-\sigma_2)\,\tau_{\theta\theta_{(2)}} - \sigma_2\,\tau_{rr_{(2)}}\right] \tag{76}$$

Integrating these equations for $u_{r_{(2)}}$ and $u_{\theta_{(2)}}$ and neglecting rigid body motions, we obtain

$$u_{r_{(2)}} = \frac{1 + \sigma_2}{E_2} \left\{ (1 - 2\sigma_2) \left[C_1(2r\ell nr - r) + 2C_2 r \right] - \frac{C_3}{r} - 2\sigma_2 C_1 r \right.$$

$$\left. - \left[-(1 - \sigma_2)\, 4\frac{C_8}{r} + 4\sigma_2 C_6 r^3 - \frac{2C_7}{r^3} + 2C_5 r \right] \cos 2\theta \right\} \tag{77}$$

$$u_{\theta_{(2)}} = \frac{1 + \sigma_2}{E_2} \left\{ 4(1 - \sigma_2)\, C_1 r\theta + \frac{1}{2}\, r \left[(12 - 8\sigma_2)\, C_6 r^2 + 4C_5 \right. \right.$$

$$\left. \left. + \frac{4C_7}{r^4} - \frac{4(1 - 2\sigma_2)C_8}{r^2} \right] \sin 2\theta \right\} \tag{78}$$

In the $u_{\theta_{(2)}}$ equations, note that to assure periodicity in θ, it is necessary that

$$C_1 = 0 \tag{79}$$

APPLICATION OF BOUNDARY AND COMPATABILITY EQUATIONS TO DETERMINE UNKNOWN COEFFICIENTS

The system of equations developed above contains nine unknowns; i.e., C_2, C_3, C_5, C_6, C_7, C_8, $\overline{\tau}_{rr_0}$, $\overline{\tau}_{rr_1}$, and $\overline{\tau}_{r\theta_1}$. Evaluation of these constants entails use of the boundary conditions at $r = a$ and the stress and displacement compatability requirements at $r = b$.

Boundary Conditions at $r = a$

$$\tau_{rr_{(2)}}\Big|_{r=a} = 2C_2 + \frac{C_3}{a^2} - \left(2C_5 + \frac{6C_7}{a^4} + \frac{4C_8}{a^2} \right) \cos 2\theta = 0$$

$$\tau_{r\theta_{(2)}}\Big|_{r=a} = \left(2C_5 + 6C_6 a^2 - \frac{6C_7}{a^4} - \frac{2C_8}{a^2} \right) \sin 2\theta = 0 \qquad (80)$$

Therefore

$$2C_2 + \frac{C_3}{a^2} = 0 \qquad (81)$$

$$2C_5 + \frac{6C_7}{a^4} + \frac{4C_8}{a^2} = 0 \qquad (82)$$

$$2C_5 + 6C_6 a^2 - \frac{6C_7}{a^4} - \frac{2C_8}{a^2} = 0 \qquad (83)$$

Stress Compatability Relations at $r = b$

$$\tau_{rr_{(2)}}\Big|_{r=b} = 2C_2 + \frac{C_3}{b^2} - \left(2C_5 + \frac{6C_7}{b^4} + \frac{4C_8}{b^2} \right) \cos 2\theta = \bar{\tau}_{rr_0} + \bar{\tau}_{rr_1} \cos 2\theta$$

$$\tau_{r\theta_{(2)}}\Big|_{r=b} = \left(2C_5 + 6C_6 b^2 - \frac{6C_7}{b^4} - \frac{2C_8}{b^2} \right) \sin 2\theta = \bar{\tau}_{r\theta_1} \sin 2\theta \qquad (84)$$

Therefore

$$2C_2 + \frac{C_3}{b^2} = \bar{\tau}_{rr_0} \tag{85}$$

$$-\left(2C_5 + \frac{6C_7}{b^4} + \frac{4C_8}{b^2}\right) = \bar{\tau}_{rr_1} \tag{86}$$

$$2C_5 + 6C_6 b^2 - \frac{6C_7}{b^4} - \frac{2C_8}{b^2} = \bar{\tau}_{r\theta} \tag{87}$$

<u>Displacement Compatability Relations at $r = b$</u>

$$\underset{r=b}{u_{r_{(2)}}} = \underset{r=b}{u_{r_{(1)}}} = \underset{r=b}{u'_{r_{(1)}}} + \underset{r=b}{\bar{u}_{r_{(1)}}}$$

$$\underset{r=b}{u_{\theta_{(2)}}} = \underset{r=b}{u_{\theta_{(1)}}} = \underset{r=b}{u'_{\theta_{(1)}}} + \underset{r=b}{\bar{u}_{\theta_{(1)}}} \tag{88}$$

Using Equations (65), (66), (69), (70), (77), and (78), there results

$$\frac{1 + \sigma_2}{E_2}\left[(1 - 2\sigma_2)(2C_2 b) - \frac{C_3}{b}\right] = - \frac{\tau_0 b}{E_1}(1 + \sigma_1) - \frac{\bar{\tau}_{rr_0} b}{E_1}(1 + \sigma_1) \tag{89}$$

$$- \frac{(1+\sigma_2)}{E_2}\left[4\sigma_2 C_6 b^3 + 2C_5 b - \frac{2C_7}{b^3} - (1-\sigma_2)\frac{4C_8}{b}\right] = - \frac{2\tau_0 b}{E_1}(1-2\sigma_1)(1+\sigma_1)$$

$$+ \frac{2(1-\sigma_1)}{E_1} b \left[-\bar{\tau}_{rr_1}\left(\frac{5}{6}-\sigma_1\right) + \bar{\tau}_{r\theta}\left(\frac{2}{3}-\sigma_1\right)\right] \tag{90}$$

$$\frac{1+\sigma_2}{E_2}\left[(6-4\sigma_2)C_6 b^3 + 2C_5 b + 2\frac{C_7}{b^3} - \frac{2(1-2\sigma_2)C_8}{b}\right] = \frac{2\tau_0 b}{E_1}\left[(1-2\sigma_1)(1+\sigma_1)\right]$$

$$- \frac{(1+\sigma_1)}{E_1} b \left[-\bar{\tau}_{rr_1}\left(\frac{4}{3}-2\sigma_1\right) + \bar{\tau}_{r\theta_1}\left(\frac{5}{3}-2\sigma_1\right)\right] \tag{91}$$

COMPUTATION AND NUMERICAL RESULTS

Equations (81), (82), (83), (85), (86), (87), (89), (90) and (91) form a system of nine equations for the nine unknowns. For compatibility with the nondimensional dynamic analysis presented previously, and to expedite computation, the following notation and assumptions are introduced:

$$\frac{1-\sigma_1}{E_1} = \frac{2}{\mu_1}, \quad \frac{1-\sigma_2}{E_2} = \frac{2}{\mu_2}, \quad \frac{\mu_2}{\mu_1} = \frac{1}{\nu}$$

$$a = 1, \quad \tau_0 = 1, \quad b = \frac{b}{a} = \eta \tag{92}$$

Equation (92) automatically correctly nondimensionalizes all the stress expressions. The nine equations then become

$$2C_2 + C_3 = 0$$

$$2C_5 + 6C_7 + 4C_8 = 0$$

$$2C_5 + 6C_6 - -6C_7 - 2C_8 = 0$$

$$2C_5 + \frac{6C_7}{\eta^4} + \frac{4C_8}{\eta^2} = - \bar{\tau}_{rr_1}$$

$$2C_2 + \frac{C_3}{\eta^2} = \bar{\tau}_{rr_0}$$

$$2C_5 + 6C_6\eta^2 - \frac{6C_7}{\eta^4} - \frac{2C_8}{\eta^2} = \bar{\tau}_{r\theta_1}$$

$$(1 - 2\sigma_2)(2C_2) - \frac{C_3}{\eta^2} = \frac{1}{\nu}(-1 - \bar{\tau}_{rr_0})$$

$$4\sigma_2 C_6\eta^2 + 2C_5 - \frac{2C_7}{\eta^4} - (1 - \sigma_2)\frac{4C_8}{\eta^2}$$

$$= \frac{1}{\nu}\left\{2(1 - 2\sigma_1) - 2\left[-\bar{\tau}_{rr_1}\left(\frac{5}{6} - \sigma_1\right) + \bar{\tau}_{r\theta_1}\left(\frac{2}{3} - \sigma_1\right)\right]\right\}$$

$$(6 - 4\sigma_2) C_6\eta^2 + 2C_5 + \frac{2C_7}{\eta^4} - \frac{2(1 - \sigma_2)C_8}{\eta^2}$$

$$= \frac{1}{\nu}\left\{2(1 - 2\sigma_1) + \left[\bar{\tau}_{rr_1}\left(\frac{4}{3} - 2\sigma_1\right) - \tau_{r\theta_1}\left(\frac{5}{3} - 2\sigma_1\right)\right]\right\} \quad (93)$$

Equation (91) is solved for $C_2 \ldots C_8$, $\bar{\tau}_{rr_0}$, $\bar{\tau}_{rr_1}$, and $\bar{\tau}_{r\theta_1}$, with $\eta = 1.05, 1.1, 1.2$ and two cases of material properties corresponding to the soft and stiff cylindrical lining of the dynamic analysis.

Soft Cylindrical Lining

$$\nu = 2.9, \ \sigma_1 = .25, \ \sigma_2 = .20$$

Stiff Cylindrical Lining

$$\nu = .31, \ \sigma_1 = .25, \ \sigma_2 = .30$$

The values of constants $C_2 \ldots C_8$, $\bar{\tau}_{rr_0}$, $\bar{\tau}_{rr_1}$, and $\bar{\tau}_{r\theta_1}$ for these two cases are tabulated in Table I.

Table I

Case I $\nu = 2.9$ $\sigma_1 = .25$ $\sigma_2 = .20$									
η	C_2	C_3	C_5	C_6	C_7	C_8	$\bar{\tau}_{rr_0}$	$\bar{\tau}_{rr_1}$	$\bar{\tau}_{r\theta_1}$
1.05	-.112	.224	.053	.002	.057	-.112	-.021	.019	.041
1.1	-.116	.232	.053	.003	.060	-.116	-.040	.033	.077
1.2	-.123	.246	.055	.004	.063	-.122	-.075	.047	.133
Case II $\nu = .31$ $\sigma_1 = .25$ $\sigma_2 = .30$									
η	C_2	C_3	C_5	C_6	C_7	C_8	$\bar{\tau}_{rr_0}$	$\bar{\tau}_{rr_1}$	$\bar{\tau}_{r\theta_1}$
1.05	-.1004	2.008	.596	-.052	.491	-1.035	-.187	.138	.298
1.1	-.903	1.806	.595	-.069	.458	- .984	-.313	.187	.442
1.2	-.775	1.551	.572	-.071	.429	- .930	-.474	.197	.578

Using the values of the constants tabulated in Table I, it is possible to evaluate the stresses and displacements at the boundaries of the stiff and soft cylinder. For purposes of this report, $\tau^*_{\theta\theta_{(2)}}\Big|_{r=a}$, $\tau^*_{\theta\theta_{(1)}}\Big|_{r=b}$, and $\tau^*_{rr_{(1)}}\Big|_{r=b}$ are calculated for each case. The (*) notation is consistant with the nondimensional notation used in the dynamic analysis.

$$\tau^*_{\theta\theta_{(2)}}\Big|_{r=a} = 2C_2 - C_3 + (2C_5 + 12C_6 + 6C_7)\cos 2\theta$$

$$\tau^*_{\theta\theta_{(1)}}\Big|_{r=b} = \tau'_{\theta\theta_{(1)}}\Big|_{r=b} + \tau_{\theta\theta_{(1)}}\Big|_{r=b} = \frac{1}{1-\sigma_1}\left[-1 + (2 - 4\sigma_1)\cos 2\theta\right] - \bar{\tau}_{rr_0}$$

$$+ (\bar{\tau}_{rr_1} - 2\bar{\tau}_{r\theta_1})\cos 2\theta \tag{94}$$

$$\bar{\tau}^*_{rr_{(1)}}\Big|_{r=b} = \bar{\tau}_{rr_0} + \bar{\tau}_{rr_1}\cos 2\theta$$

These values for $\theta = \pi, \pi/2$ and $\eta = 1.05, 1.1, 1.2$ are plotted on Figs. 3, 4, 5, 6, 8, 9, 10 and 11 at $\alpha_1 a = 0$; this represents the infinite wave length or static solution.

REFERENCES

1. Pao, Yih Hsing. "Dynamical Stress Concentration in an Elastic Plate," ASME Paper No. 61-APMW-17, August 1961.

2. Pao, Yih Hsing and Mow, C. C. "Dynamic Stress Concentration in an Elastic Plate With Rigid Circular Inclusions," MITRE SR-41, December 1961; see also Proceedings of 4th U.S. National Congress of Applied Mechanics, 18 June 1962.

3. Baron, M. L., Bleich, H. H. and Weidlinger, P. Theoretical Studies on Ground Shock Phenomena, ESD-TDR-63-425.

4. Nishimura, G. and Jimbo, Y. "A Dynamical Problem of Stress Concentration," Journal of the Faculty of Engineering, University of Tokyo, Japan, Vol. 24, 1955, p. 101.

5. Junger, M. "Sound Scattering by Thin Shell," The Journal of Acoustical Society of America, Vol. 24, No. 2, July 1952, p. 366.

6. Mindlin, R. D. and Bleich, H. H. "Response of an Elastic Cylindrical Shell to a Transverse Step Shock Wave," Journal of Applied Mechanics, June 1953, p. 189.

7. Baron, M. L. and Parnes, R. Diffraction of a Pressure Wave by an Elastically Lined Cylindrical Cavity in an Elastic Medium, ESD-TDR-63-384.

8. Wang, C. T. Applied Elasticity, McGraw-Hill Book Company, New York, 1953.

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* The MITRE Corporation Bedford, Massachusetts	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP

3. REPORT TITLE

Dynamic Stresses in an Elastic Cylindrical Lining of Arbitrary Thickness in an Elastic Medium

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

N. A.

5. AUTHOR(S) *(Last name, first name, initial)*

Mow, C. C.
McCabe, Warren L.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
April 1965	54	8

8a. CONTRACT OR GRANT NO. AF33(600)39852 b. PROJECT NO. 607 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) ESD-TR-65-112 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* SR-57

10. AVAILABILITY/LIMITATION NOTICES

Qualified requestors may obtain from DDC. DDC release to OTS authorized.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Deputy for Advanced Planning, Electronic Systems Division, L. G. Hanscom Field, Bedford, Massachusetts

13. ABSTRACT

Dynamic stresses in a thick-wall elastic cylinder in an infinite elastic medium during passage of plane, compressional waves are investigated. Dynamic stresses around the cylinder in the elastic medium are also determined. Numerical results for two different cylinders with ratios of outer radius to inner radius ranging from 1.05 to 1.20 are presented in a dimensionless form. It is shown that increasing thickness does not, in general, reduce stresses in the cylinder; in addition, dynamic stresses at certain wave numbers are higher than the corresponding static value.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
DYNAMIC STRESSES						
Elasticity						
Engineering Mechanics						
Structure Interaction						
Wave Interaction						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
__________________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
__________________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
__________________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.